Measures on perfect PAC fields with pro-cyclic Galois group

I. Halupczok

Ecole Normale Superieure
rue d'Ulm, Paris

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Introduction

- Emerged from work group in Lyon (France) with Thomas Blossier, Amador Martin Pizarro and Juan Pons Llopis.
- Answer a question of Hrushovski on “measures on fields with bounded Galois groups”.

Outline

1. Measures on pseudo-finite fields
2. How to generalize?
3. Results

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2. How to generalize?

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Situation for pseudo-finite fields

- Let K be a pseudo-finite field.
- Want to understand definable sets $X \subset K^n$.
- Method: Associate invariants to X:
  - Dimension $\dim(X)$
  - Measure $\mu(X)$

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Dimension

Definition (Dimension)

View $X \subset K^n$ as a subset of $\tilde{K}^n$. Set $\dim(X) :=$ dimension of Zariski-closure of X in $\tilde{K}^n$.

Example: X absolute irreducible variety

- K is PAC $\Rightarrow X(K)$ is dense in $X(\tilde{K})$.
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- $\dim X = 1$
- "half of the elements" of K are squares.
- $\rightsquigarrow \mu(X) = \frac{1}{2}$.

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For definition of μ need almost-quantifier-elimination:

Theorem

K PSF field, X definable. $\Rightarrow X = \pi(V(K))$, *where*

- *V is algebraic*
- *π is a projection*
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Definition of $\mu(X)$:

- $X \subset K^n$ algebraic set $\rightsquigarrow \mu(X) :=$ number of irreducible components of Zariski closure of X in $\tilde{K}$
- X not algebraic:
  - Almost-qe $\rightsquigarrow \pi\colon V \to X$
  - If all fibers have size $n \rightsquigarrow \mu(X) := \frac{1}{n}\mu(V)$.
  - Otherwise: A bit more work. (Best way: use Galois stratifications.)

Example: $X = \{\text{squares}\}$:

- $\{\text{squares}\} = \pi(V(y^2 = x))$ where $\pi =$ proj. on x-coordinate
- $V := V(y^2 = x)$ is absolutely irreducible $\to \mu(V) = 1$.
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Properties

What does μ satisfy?

- $\mu(V) = 1$ if V absolutely irreducible.
- $\mu(X \dot\cup Y) = \begin{cases} \mu(X) + \mu(Y) & \text{if } \dim X = \dim Y \\ \mu(X) & \text{if } \dim X > \dim Y \end{cases}$
- $\mu(X \times Y) = \mu(X) \cdot \mu(Y)$
- $f : X \to Y$ definable bijection $\Rightarrow \mu(X) = \mu(Y)$
- More generally: Fubini: $f : X \to Y$ definable, surjective, with constant fiber size $\Rightarrow \mu(X) = \mu(Y) \cdot \mu(\text{fiber})$.

Theorem (Hrushovski)

These properties axiomatize the measure.

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Outline

Generalize: To which fields?

Generalize the measure to other fields. Which fields?

- Need control over definable sets.
- Recall: K PSF $\iff$ K PAC, perfect and $\mathrm{Gal}_K := \mathrm{Gal}(\tilde{K}/K) = \hat{\mathbb{Z}}$.

Theorem (Hrushovski)

Almost-qe still holds if K PAC, perfect and Gal_K bounded.

- Gal_K bounded := has only a finite number of quotients of each fixed cardinality

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Generalize the measure to other fields. Which fields?

- Need control over definable sets.
- Recall: K PSF $\iff$ K PAC, perfect and $\mathrm{Gal}_K := \mathrm{Gal}(\tilde{K}/K) = \hat{\mathbb{Z}}$.

Theorem (Hrushovski)

Almost-qe still holds if K PAC, perfect and Gal_K bounded.

- Gal_K bounded := has only a finite number of quotients of each fixed cardinality

Question (Hrushovski)

Does a measure exist if K PAC, perfect and Gal_K bounded?

Generalized axioms

Can we expect all the axioms?

- Fubini: $f : X \to Y$ definable, surjective, with constant fiber size $\Rightarrow \mu(X) = \mu(Y) \cdot \mu(\text{fiber})$.

Example: $K = \tilde{K}$, $f : K^\times \to K^\times, x \mapsto x^2$.
Fibers have 2 elements, but left and right set are the same. ↯

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This works for $K = \tilde{K}$:

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Outline

Result

Let K be PAC, perfect and Gal_K bounded.

Theorem (H.)

1. *If Gal_K is pro-cyclic, it works (μ exists).*
2. *With one (natural) additional assumption, it is even unique.*
3. *For general bounded Gal_K (e.g. $\mathrm{Gal}_K = \hat{\mathbb{Z}} * \hat{\mathbb{Z}}$), no measure exists.*

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Why only for pro-cyclic Galois groups

- Does not work in general because there are many definable bijections:
  Have more or less $X \xrightarrow{1:1} Y$ for $Y \subsetneq X$.
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  - It works for PSF fields and...
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When Gal_K pro-cyclic

Example: Consider $X = \{\text{squares}\}$.

- If K is PSF, then $K = \{\text{squares}\} \,\dot\cup\, \{\text{non-squares}\}$.
  $\mu_K(\{\text{squares}\}) = \mu_K(\{\text{non-squares}\}) = \frac{1}{2}$.
- Now add all square roots to K: $L := K \cup \sqrt{K} \cup \sqrt{\sqrt{K}} \cup \ldots$.
- $\rightsquigarrow$ All non-squares also become squares.
- $\rightsquigarrow$ $\mu_L(\{\text{squares}\}) = \mu_K(\{\text{squares}\}) + \mu_K(\{\text{non-squares}\}) = 1$

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General idea is the same.

- Pretend that K is an extension of a PSF field.
- Which definable sets in the PSF field become part of X when extending to K?
- Set $\mu(X)$ to be the sum of the measures of those sets.
- Real way to do this is using Galois stratification.

Remark: If one tries to define μ for K directly (without going through the PSF-measure), it does not work.

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Example: K is PSF and $X = \{\text{squares}\}$.

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