Trees of varieties over $\mathbb{Z}_p$

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Outline

1. Goal
2. Understanding the trees
3. Definition of complexity d trees

Trees of sets in $\mathbb{Z}_p^n$

$X \subset \mathbb{Z}_p^n$ yields a tree $\mathrm{T}(X)$:

- $\lambda \in \mathbb{N} \rightsquigarrow$ consider all balls of "radius" λ intersecting X:

$$X_\lambda := \{B = \bar{a} + p^\lambda \mathbb{Z}_p^n \mid B \cap X \neq \emptyset\}$$

- $\mathrm{T}(X) := \dot{\bigcup}_\lambda X_\lambda$
- Inclusion of balls induces tree structure

Examples:

- $X = \mathbb{Z}_p \rightsquigarrow$ Every node of $\mathrm{T}(X)$ has p children
- X finite $\rightsquigarrow$ each $x \in X$ corresponds to infinite path in $\mathrm{T}(X)$: $\bar{x} + \mathbb{Z}_p^n \supset \bar{x} + p\mathbb{Z}_p^n \supset \bar{x} + p^2\mathbb{Z}_p^n \supset \dots$
  Paths of $\bar{x}$ and $\bar{x}'$ separate at depth $\min_i v(x_i - x_i')$

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Goal (1)

Goal: describe which trees $\mathrm{T}(X)$ one can get if X is. . . :

- $X = \{\bar{x} \mid f_1(\bar{x}) = \cdots = f_k(\bar{x}) = 0\}$ affine algebraic set.
- More generally: X definable by first order formula in the valued field language.

Definition (Scowcroft, van den Dries)

X definable, $\dim X :=$ dimension of Zariski closure of X in $\tilde{\mathbb{Q}}_p$.

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We will define *trees of complexity d*.

Conjecture (H.)

X definable, $\dim X = d \Rightarrow \mathrm{T}(X)$ *is of complexity d.*

Goal of remainder of talk: make definition of trees of complexity d plausible.
The other direction is true:

Theorem (H.)

$\mathcal{T}$ tree of complexity $d \Rightarrow$ there exists a definable X such that $\mathrm{T}(X) = \mathcal{T}$

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Motivation: poincaré series

- $X \subset \mathbb{Z}_p^n \rightsquigarrow$ Poincaré series of X:

$$P_X(Z) := \sum_{\lambda \geq 0} \#X_\lambda \cdot Z^\lambda \in \mathbb{Z}[[Z]]$$

(Recall: $X_\lambda =$ nodes of $\mathrm{T}(X)$ at depth λ.)

Theorem (Denef)

X definable $\Rightarrow P_X(Z) \in \mathbb{Q}(Z)$.

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- Indeed, complexity d trees have rational Poincaré series.

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Lemma

$X, X' \subset \mathbb{Z}_p^n$ p-adically closed. Then:

$$\{\textit{bijective isometries } X \to X'\} \overset{1:1}{\leftrightarrow} \{\textit{isomorphisms } T(X) \to T(X')\}$$

- So: trees help understanding sets up to isometry
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Outline

Key lemma

Crucial ingredient is (a generalization of):

Lemma (Key lemma)

Suppose $\phi\colon \mathbb{Z}_p \to \mathbb{Z}_p$ *satisfies* $v(\phi(x') - \phi(x)) \geq v(x' - x)$.
Then $\mathrm{T}(\mathrm{graph}(\phi)) \cong \mathrm{T}(\mathrm{graph}(x \mapsto 0)) \cong \mathrm{T}(\mathbb{Z}_p)$

Smooth plane curves (1)

Suppose X is smooth plane curve.

- For each $(x_0, y_0) \in X$: implicit function theorem yields ball $(x_0, y_0) + p^\lambda \mathbb{Z}_p^2$ on which X is the graph of a function ϕ
- If $v(\phi'(x_0)) < 0$ then exchange coordinates $\rightsquigarrow v(\phi'(x_0)) \geq 0$
- $\phi'(x_0) \approx \frac{\phi(x)-\phi(x_0)}{x-x_0}$
- On smaller ball: $v(\phi(x) - \phi(x')) \geq v(x - x')$
- Key lemma $\Rightarrow$ $\mathrm{T}(X)$ on $(x_0, y_0) + p^\lambda \mathbb{Z}_p^2$ is isomorphic to $\mathrm{T}(\mathbb{Z}_p)$.

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Smooth plane curves (2)

- $\mathbb{Z}_p^2$ compact, X closed in $\mathbb{Z}_p^2 \Rightarrow X$ is covered by finitely many balls B_i on which the tree is $\mathrm{T}(\mathbb{Z}_p)$.
- We may suppose that the B_i are disjoint.
- Total tree of X is:
  - finite tree with leafs B_i
  - a copy of $\mathrm{T}(\mathbb{Z}_p)$ attached to each leaf.

Arbitrary smooth algebraic sets work similarly.

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X is cusp curve

Example: $X = \{(x,y) \in \mathbb{Z}_p^2 \mid x^3 = y^2\}$, $p \neq 2$

- $T(X)$ contains $\{p^\lambda \mathbb{Z}_p^2 \mid \lambda \geq 0\}$. What are the side branches?
- $x^3 = y^2$ (and suppose $\lambda := v(x) > 0$) $\Rightarrow$
  - $v(y) = \frac{3}{2} v(x) > \lambda$
  - $y = \pm x\sqrt{x}$, i.e. x is square
    $\iff$ $2|\lambda$ and $\mathrm{ac}(x)$ is quare in $\mathbb{F}_p$
- $(x,y) \in B := (p^\lambda x_0, 0) + p^{\lambda+1}\mathbb{Z}_p^2$ with $x_0 \in \mathbb{Z}_p^\times$
  B is child of $p^\lambda \mathbb{Z}_p^2$.
- The tree on B:
  - $X \cap B =$ union of the two graphs $x \mapsto \pm x\sqrt{x}$
  - Distance between graphs is $\frac{3}{2}\lambda$
  - Satisfy (generalization of) key lemma
  - $\Rightarrow$ Tree on B is
    $\mathrm{T}(\mathbb{Z}_p) \times \{$Two paths separating at depth $\frac{1}{2}\lambda - 1\}$
- $\rightsquigarrow$ Total tree: $\frac{p-1}{2}$ such side branches at even depths, no side branches at odd depths

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Outline

1. Goal
2. Understanding the trees
3. Definition of complexity d trees

Generalizing the idea: trees of complexity 0

Examples $\rightsquigarrow$ definition of trees of complexity d

Trees of complexity 0:

- $\dim X = 0 \iff X$ finite
  $\rightsquigarrow$ *trees of complexity* 0 := tree with finitely many bifurcations

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Cusp $\rightsquigarrow$ *Tree of complexity* 1 :=

- Tree $\mathcal{T}_0$ with finitely many bifurcations (corresponding to singularities of X)
  [Cusp: only one path to 0]
- $+$ additional side branches at each node v
  [Cusp: $\frac{p-1}{2}$ side branches if $2 \mid \mathrm{depth}(v)$]
- Side branches are of the form $\mathrm{T}(\mathbb{Z}_p) \times \mathcal{T}_v$
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  [Cusp: $\mathcal{T}_v$ two paths separating at depth $\frac{1}{2}\,\mathrm{depth}(v) - 1$]
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Definition of trees of complexity d

- General definition:
  - Define *uniform families of trees of complexity d*
  - Trees of complexity $d+1$ are with finitely many bifurcations + uniform side branches of complexity d

Theorem (H.)

For any definable $X \subset \mathbb{Z}_p^2$, $\mathrm{T}(X)$ is of complexity $\dim X$.

- (Trees of definable set are not really more complicated than trees of varieties.)

Idea of proof:

- For varities: similar to cusp (use theorem of Puiseux)
- For definable sets: additionally use cell decomposition

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