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Dimension in topological structures: topological closure and local property

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Abstract. Let K be a first-order structure with a dimension function, satisfying some natural conditions. Let A be a definable set. If every point in A has a definable neighborhood in A with dimension less than p, then A has dimension less than p.

Contents

1. Introduction

Let $\mathbb{K}$ be a first-order topological structure without isolated points; topological structure means that we have a topology on $\mathbb{K}$ which has a basis consisting of a $\emptyset$-definable family of subsets of $\mathbb{K}$ (see [Pil87]). By "definable", unless otherwise specified, we will mean "definable in $\mathbb{K}$ using parameters". Let dim be a **dimension** on $\mathbb{K}$, that is a function mapping each nonempty definable set to some natural number and satisfying some axioms.

We show that, under certain extra assumptions, the dimension satisfies two important properties: for every definable set A,

(1) $\dim(\overline{A}) = \dim(A)$ (where $\overline{A}$ is the topological closure of A);
(2) If, for every $a \in A$, there exists U definable neighborhood of a, such that $\dim(A \cap U) \leq p$, then $\dim(A) \leq p$.

It turns out that (1) is easy, whereas (2) really seems to need some work.

We now list the assumptions on the dimension. We set $\dim(\emptyset) = -\infty$. Let A, A' definable subsets of $\mathbb{K}^n$ and B be a definable subset of $\mathbb{K}^{n+1}$. If $a \in \mathbb{K}^n$, we

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write $B_a = \{b \in \mathbb{K} \mid (a, b) \in B\}$ for the fiber. Moreover, we write $\Pi^m_p : \mathbb{K}^m \to \mathbb{K}^p$ for the projection onto the first p coordinates.

(Dim 1) $\dim(\mathbb{K}) = 1$, $\dim(\{a\}) = 0$ for every $a \in \mathbb{K}$.
(Dim 2) $\dim(A \cup A') = \max(\dim(A), \dim(A'))$.
(Dim 3) For every permutation σ of coordinates, $\dim(A^\sigma) = \dim(A)$.
(Dim 4) Let $B(0) \coloneqq \{a \in \mathbb{K}^n : \dim(B_a) = 0\}$. Then, $B(0)$ is definable, with the same parameters as B.
(Dim 5) Let i be either 0 or 1. Let $C \coloneqq \Pi^{n+1}_n(B)$. Assume that, for every $a \in C$, $\dim(B_a) = i$. Then, $\dim(B) = \dim(C) + i$.

The following axiom relates the dimension to the topology.

(TD 1) If A is open (in $\mathbb{K}^n$), then $\dim(A) = n$.

We will also need the following condition.

(Group) $\mathbb{K}$ expands an Abelian group $\langle K, +, 0\rangle$; moreover, this group is a Hausdorff topological group.

Axioms (Dim 1–5) are almost the same as the axioms for dimension in [vdD89]. The only difference is that in the analogue of Axiom (Dim 3), van den Dries does not require that one can use the parameters of definition of B to define $B(0)$ (but see [For11a, §4] and [vdD89, Proposition 1.7]).

In topological structures, a natural way to define dimension is the following:

(TD 2) For $A \subset \mathbb{K}^n$, let $\dim(A)$ be the largest integer p, such that, after a permutation of coordinates, $\Pi^n_p(A)$ has nonempty interior in $\mathbb{K}^p$.

Notice that this definition implies Axioms (TD 1), (Dim 3), (Dim 4), and, since $\mathbb{K}$ has no isolated points, (Dim 1). Thus in that case, the only interesting axioms are (Dim 2) and (Dim 5). However, in Theorem 3.1 below, we do not need dimension to be defined that way, so let us simply treat (TD 2) as another axiom.

See §4 for some examples of structures satisfying various subsets of the axioms.

An easy but important remark is the following.

Remark 1.1. Let $A \subseteq \mathbb{K}^n$ be definable.

(1) Assume that Axiom (TD 2) holds. Then, $\dim(A) = n$ iff A has nonempty interior (in $\mathbb{K}^n$).
(2) Assume that Axiom (Dim 2) holds. If $A \subseteq B$, then $\dim(A) \le \dim(B)$.

2. Closure

Proposition 2.1. *Assume that Axioms (TD 2) and (Dim 2) hold. Then, for every definable set $A \subseteq \mathbb{K}^n$, $\dim(\overline{A}) = \dim(A)$.*

Proof. Let $p \coloneqq \dim(\overline{A})$.

Case 1: $p = n$. Define $\partial A \coloneqq \overline{A} \setminus A$. Notice that $\overline{A} = A \cup \partial A$ and that ∂A has empty interior, and therefore $\dim(\partial A) < n$. If, for a contradiction, $\dim(A) < n$, then, by (Dim 2), $\dim(\overline{A}) < n$, absurd.

Case 2: p any. After a permutation of coordinates, w.l.o.g. $\Pi^n_p(\overline{A})$ has dimension p. Let $B \coloneqq \Pi^n_p(A)$. If, for a contradiction, $\dim(A) < p$, then $\dim(B) < p$. Notice that $\Pi^n_p(\overline{A}) \subseteq \overline{B}$. Thus, by Case 1, $\dim(\overline{B}) = \dim(B) < p$, absurd. ☐

3. Local property

Theorem 3.1. *Assume that Axioms (TD 1), (Dim 1–5), and (Group) hold. Let* $A \subseteq \mathbb{K}^n$ *be definable. Assume that, for every* $a \in A$*, there exists a definable neighborhood* U *of* a*, such that* $\dim(A \cap U) \leq p$*. Then,* $\dim(A) \leq p$*.*

The proof is quite more difficult than the proof of Proposition 2.1. It might be surprising that (Group) is needed; in Section 4, we will see a counter-example without (Group).

For all this section, we assume that Axioms (TD 1), (Dim 1–5), and (Group) hold.

We make heavy use of the results in [For11a, §1–4]. We first need the following definitions.

Definition 3.2. Let $a \in \mathbb{K}^n$ and $C \subseteq \mathbb{K}$. The **rank** of a over C, denoted by $\operatorname{rk}(a/C)$, is the minimum p such that there exists a set $X \subseteq \mathbb{K}^n$ which is definable with parameters from C, with $a \in X$ and $\dim(X) = p$.

Notice that if $a \in \mathbb{K}$, then $\operatorname{rk}(a/C) \leq 1$.

Definition 3.3. Let A, B, C be subsets of $\mathbb{K}$. We define the preindependence relation $A \mathop{\smile\!\!\!\!\!\!\mid}_B C$ iff, for every finite subset $\bar{a} \subseteq A$, $\operatorname{rk}(\bar{a}/B) = \operatorname{rk}(\bar{a}/BC)$.

Lemma 3.4. $\mathop{\smile\!\!\!\!\!\!\mid}$ *is symmetric: that is,* $A \mathop{\smile\!\!\!\!\!\!\mid}_B C$ *iff* $C \mathop{\smile\!\!\!\!\!\!\mid}_B A$*.*

Proof. The following proof is a standard exercise (see [For11a, Theorem 4.3 and Lemma 3.6]). Let us assume that $A \mathop{\smile\!\!\!\!\!\!\mid}_B C$; we have to prove that $C \mathop{\smile\!\!\!\!\!\!\mid}_B A$; thus, we have to show that, for every finite $C' \subseteq C$, $\operatorname{rk}(C'/BA) = \operatorname{rk}(C'/B)$. Since C' is finite, there exists a finite set $A' \subseteq A$, such that $\operatorname{rk}(C'/BA) = \operatorname{rk}(C'/BA')$. By assumption, we have $\operatorname{rk}(A'/BC) = \operatorname{rk}(A'/B)$. Since $\operatorname{rk}(A'/BC) \leq \operatorname{rk}(A'/BC') \leq \operatorname{rk}(A'/B)$, we also have $\operatorname{rk}(A'/BC') = \operatorname{rk}(A'/B)$. Now we can use the additivity of rank:

$$\operatorname{rk}(A'/BC') + \operatorname{rk}(C'/B) = \operatorname{rk}(A'C'/B) = \operatorname{rk}(C'/BA') + \operatorname{rk}(A'/B).$$

Since all the summands in the above equality are finite, $\operatorname{rk}(A'/BC') = \operatorname{rk}(A'/B)$ implies $\operatorname{rk}(C'/B) = \operatorname{rk}(C'/BA') = \operatorname{rk}(C'/BA)$. □

Let κ be some big cardinal. From now on, we assume that $\mathbb{K}$ is a monster model (since it suffices to prove the theorem in some $\mathbb{K}'$ elementary extension of $\mathbb{K}$), that is a κ-saturated and κ-homogeneous structure. By "small", we will mean of cardinality less than κ. Moreover, we assume that A is definable without parameters (otherwise, we add the parameters of definition of A to the language).

The following fact follows easily from the definitions.

Fact 3.5 ([For11a, Lemma 3.69])**.** *Let* $A \subseteq \mathbb{K}^n$ *be definable without parameters, and let* $\big(U_t : t \in T\big)$ *be a family of subsets of* $\mathbb{K}^n$*, such that each* U_t *is definable with parameters* t*. Let* $p \leq n$*, and assume that, for every* $a \in A$*, there exists* $t \in T$*, such that* $a \in U_t$*,* $a \mathop{\smile\!\!\!\!\!\!\mid} t$*, and* $\dim(A \cap U_t) \leq p$*. Then,* $\dim(A) \leq p$*.*

We want to apply the above result. Let A be as in Theorem 3.1 and let $a \in A$. By the theorem's assumption, there exists a definable set V which is a neighborhood of a and such that $\dim(A \cap V) \leq p$. If we could find a definable set $V' \subseteq V$ and some parameter t such that V' is definable over t, $a \in V'$, and $a \mathop{\smile\!\!\!\!\!\!\mid} t$, we could then apply Fact 3.5 to conclude. The remainder of the proof is finding such V'.

Since $\mathbb{K}$ is a first-order topological structure, there exists a definable family $\mathcal{B} := \big(B_t : t \in I\big)$ of subsets of $\mathbb{K}$, definable without parameters, such that $\mathcal{B}$ is a basis for the topology of $\mathbb{K}$. Given $b \in I^n$, we denote by B_b the set $B_{b_1} \times \ldots B_{b_n}$. Given $c \in \mathbb{K}^n$ and $E, E' \subseteq \mathbb{K}^n$, we denote by $E + c$ the translation of E by c and by $E + E' := \{e + e' : e \in E, e' \in E'\}$.

Notice that $\big(B_b : b \in I^n\big)$ is a basis for the topology of $\mathbb{K}^n$.

Lemma 3.6 (Cf. [For11a, Lemma 9.14])**.** *Let $d \in \mathbb{K}^n$, V be a definable neighborhood of d, and C be a small subset of $\mathbb{K}$. Then, there exists $b \in I^n$ such that $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_d C$ and $d \in B_b \subseteq V$.*

Proof. Let $J := \{b \in I^n : d \in B_b\}$. Let $\leq$ be the quasi-ordering on J given by reverse inclusion: $b \leq b'$ iff $B_b \supseteq B_{b'}$. Fix $e \in J$ such that $B_e \subseteq V$. Since $\langle J, \leq\rangle$ is a directed quasi-order definable over d, [For11a, Lemma 3.68] implies that there exists $b \in J$ such that $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_d C$ and $b \geq e$. $\square$

Notice that if in the above lemma we were able to reach the conclusion that $e \mathop{\smile\!\!\!\!\!\!\mid} D$ instead of $e \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_d D$, we would be done.

Lemma 3.7 (Cf. [For11a, Lemma 9.18])**.** *Let $a \in \mathbb{K}^n$, V be a definable neighborhood of a, and C be a small subset of $\mathbb{K}$. Then, there exist $e \in \mathbb{K}^n$ and $b \in I^n$ such that $a \in B_b + e \subseteq V$ and $eb \mathop{\smile\!\!\!\!\!\!\mid} C$.*

Proof. Let $V_0 := V - a$; it is a definable neighborhood of 0. Since $\mathbb{K}^n$ is a topological group, there exists V_1 definable open neighborhood of 0, such that $V_1 = -V_1$ and $V_1 + V_1 \subseteq V_0$. By Lemma 3.6, applied to $d = 0$, there exists $b \in I^n$ such that $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_0 C$ and $0 \in B_b \subseteq V_1$; since 0 is in the definable closure of the empty set, we have $b \mathop{\smile\!\!\!\!\!\!\mid} C$. Let $W := a - B_b$. Notice that W is an open subset of $\mathbb{K}^n$; thus, $\dim(W) = n$, and, by monstrosity of $\mathbb{K}$, there exists $e \in W$ such that $\operatorname{rk}(e/Cab) = n$ [For11a, Remark 3.10], and therefore $e \mathop{\smile\!\!\!\!\!\!\mid} Cab$. We have to show that b and e satisfy the conclusion.

Claim 1. $a \in B_b + e$

It follows immediately form $e \in a - B_b$.

Claim 2. $B_b + e \subseteq V$.

Let $f \in B_b + e$: we need to prove that $f \in V$. We know that $e \in W = a - B_b$, and therefore $e - a \in -B_b \subseteq V_1$. Since $f - e \in B_b \subseteq V_1$, we have $f - a \in V_1 + V_1 \subseteq V - a$, and therefore $f \in V$.

Claim 3. $eb \mathop{\smile\!\!\!\!\!\!\mid} C$.

We have that $e \mathop{\smile\!\!\!\!\!\!\mid} Cab$ implies $e \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_b Cab$, which implies $eb \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_b Ca$, and hence $eb \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_b C$. Together with $b \mathop{\smile\!\!\!\!\!\!\mid} C$, by transitivity the above implies $eb \mathop{\smile\!\!\!\!\!\!\mid} C$. $\square$

Finally, we are in a position to apply Fact 3.5. Let V an open neighborhood of a, such that $\dim(V \cap A) \leq p$. Consider the definable family $\big(B_b + e : b \in I^n, e \in \mathbb{K}^n\big)$. Each set $B_b + e$ is definable with parameters be. By Lemma 3.7, applied with $C := \{a\}$, there exist $e \in \mathbb{K}^n$ and $b \in I^n$ such that $a \in B_b + e \subseteq V$ and $a \mathop{\smile\!\!\!\!\!\!\mid} eb$. Since $B_b + e \subseteq V$, we have $\dim((B_b + e) \cap A) \leq p$. Thus, the assumptions of Fact 3.5 are satisfied (for the family of sets of the form $U_{b,e} := B_b + e$), and the theorem is proved. $\square$

4. Some examples

Here is another axiom one could have required:

(Finite) A definable set of dimension 0 is finite.

In the following examples, if we do not specify the dimension, we mean the one defined by (TD 2).

- Axioms (TD 1–2), (Dim 1–5), (Group), and (Finite) hold in arbitrary o-minimal structures expanding an ordered group.
- Axioms (Dim 1–5) hold for the dimension in b-minimal structures $\mathbb{K}$ in the sense of [CL07] (by Theorems 4.2 and 4.3 in loc. sit.). Let us say that $\mathbb{K}$ is a "topological b-minimal structure" if it is b-minimal and it has a definable topology such that a set has non-empty interior if and only if it contains a ball. A topological b-minimal structure additionally satisfies (TD 2) (and hence also (TD 1)). In fact, all standard examples of b-minimal structures are topological b-minimal structures, but in general, the family of balls does not even determine the topology; see the last example below.

  Any o-minimal structure is a topological b-minimal structure. Other standard examples are Henselian valued fields of characteristic 0; these fields also satisfy (Group) and (Finite). By [CL11], this is still true if one adds an analytic structure in the sense of [CL11] to the language. By [Yin10], another example of a topological b-minimal structure is an algebraically closed valued field of residue characteristic 0 with a section added to the language, either from the residue field to the valued field or from the "leading term structure" RV to the valued field. In this example, we also have (Group), but (Finite) is not satisfied (since the image of the section is 0-dimensional).
- A d-minimal expansion of the real field satisfies (TD 1–2), (Dim 1–5), and (Group), but not (Finite) (unless it is o-minimal): see [For10a]. An example of a d-minimal expansion is the reals with a predicate for the powers of 2. (In fact, this example is also b-minimal. Question: Does d-minimality imply b-minimality?)
- Let $\mathbb{B}$ be an o-minimal structure expanding a field, and $\mathbb{A}$ a topologically dense proper elementary substructure. Then, $\langle \mathbb{B}, \mathbb{A} \rangle$ has a unique dimension function satisfying (TD 1), (Dim 1–5) and (Group), but neither (TD 2) nor (Finite); moreover, the conclusion of Proposition 2.1 does not hold in $\langle \mathbb{B}, \mathbb{A} \rangle$: see [For10b].
- Let $\mathbb{K}$ be an expansion of the real field by a closed set $E \subset \mathbb{R}$, such that $\langle \mathbb{K}, E \rangle$ does not define the natural number. Let dim be the dimension defined using Axiom (TD 2). Then, dim satisfies Axioms (Dim 2) and (Group), and, in general, will not satisfy (Finite). Moreover, dim coincides with the Hausdorff dimension. It is an open question if dim satisfies Axiom (Dim 5): see [For11b].
- The Sorgenfrey plane shows that Axiom (Group) is necessary in Theorem 3.1. More precisely, let $\mathbb{K} := \langle \mathbb{R}, +, < \rangle$, or any o-minimal expansion of $\langle \mathbb{R}, +, < \rangle$. Let dim be the usual o-minimal dimension on $\mathbb{K}$. We endow $\mathbb{K}$ with, instead of the usual order topology, the right half-open interval topology: it is the topology generated by the basis of all half-open intervals $[a, b)$, where a and b are real numbers. Notice that the half-open interval

topology is finer than the order topology; however, a subset of $\mathbb{R}^n$ has nonempty interior in the (product of the) half-open interval topology iff it has nonempty interior in the Euclidean topology. Thus, $\mathbb{K}$ satisfies Axioms (TD 1–2), (Dim 1–5), and (Finite), and moreover $\mathbb{K}$ is Hausdorff. However, $\mathbb{K}$ does not satisfy Axiom (Group), because the function minus is not continuous. Let $A \coloneqq \{\langle x, y\rangle \in \mathbb{K}^2 : x = -y\}$ be the graph of minus: the set A as dimension 1, but, with the induced topology, it is a discrete set: for every point $a = \langle c, -c\rangle \in A$, the set $U_a \coloneqq [c, c+1) \times [-c, -c+1)$ is an open neighborhood of a, such that $U_a \cap A = \{a\}$, and thus $\dim(U_a \cap A) = 0$; thus, the conclusion of Theorem 3.1 does not hold in $\mathbb{K}$.

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