MOTIVES FOR SOME PAC FIELDS

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1. Introduction

1.1. **Goal.** Given a theory T and two definable sets X and Y of T, a natural question is whether there exists a definable bijection $X \overset{1:1}{\to} Y$. "definable set of T" means: formula without parameters, up to equivalence modulo T. By "existence of a bijection" we mean: there is a formula ϕ (without parameters) such that T implies that ϕ defines the graph of a bijection $X \overset{1:1}{\to} Y$.

Proving that there is a bijection between two sets is easy: just write down the bijection. Showing that there is no bijection is more difficult. One standard technique is to use *invariants*.

The goal of this talk is to describe good invariants for the theory of pseudo-finite fields and some similar theories. First we will remind the basic definitions, in particular concerning Grothendieck rings and invariants. Then we will describe a very good invariant for pseudo-finite fields defined by Denef and Loeser, and finally, we will present a theorem which allows us to generalize this to other fields and to get additional invariants for pseudo-finite fields.

1.2. **Reminder: pseudo-finite fields.** The theory of pseudo-finite fields is the intersection of the theories of all finite fields, together with axioms saying that the field is infinite:

$$\mathrm{PSF} := \bigcap_q \mathrm{Th}(\mathbb{F}_q) \cup \{\text{there are more than } n \text{ elements} \mid n \in \mathbb{N}\}$$

In other words: a sentence is true in all pseudo-finite fields if and only if it is true in all sufficiently large finite fields.

Fact 1. *A field K is pseudo-finite if and only if K is perfect and pseudo-algebraically closed (PAC for short) and if its absolute Galois group* $\mathrm{Gal}(\tilde{K}/K)$ *is* $\hat{\mathbb{Z}}$*.*

(For this talk, it is not necessary to know what pseudo-algebraically closed means.)

2. Grothendieck rings and invariants

Definition 2. *The Grothendieck ring of a theory T is defined as follow: As an abelian group, it is:*

$$K_0(T) := \langle [X] \mid X \text{ definable} \rangle_{\mathbb{Z}} \Big/ \begin{array}{l} [X] = [Y] \quad \text{if } X \overset{1:1}{\to} Y \text{ exists} \\ [X] + [Y] = [X \mathbin{\dot\cup} Y] \end{array}$$

The ring structure on $K_0(T)$ is defined by $[X] \cdot [Y] := [X \times Y]$.

An invariant is just any map $f\colon K_0(T) \to R$, where R may be any set. However, for f to be useful, we would like to have:

- Given $X \in K_0(T)$, we can compute $f(X)$.
- We know R well; in particular we can decide whether two elements in R are equal.
- f is as injective as possible (so that we can distinguish many different definable sets).
- It is helpful if f has some additional nice properties. For example, often R will be a ring and f a ring homomorphism.

Some example invariants for the theory PSF of pseudo-finite fields.

- Fix any pseudo-finite field K. Then one has a notion of *dimension* of a definable set: embed $X(K) \subset K^n \subset \tilde{K}^n$ into the algebraic closure of K, take the Zariski closure of X in there, and take the dimension of that: $\dim_K(X) := \dim \overline{X(K)}$. (For a proof that this indeed defines an invariant, see e.g. [3].)

  Using this invariant, we can prove: $\mathbb{A}^1 \overset{1:1}{\not\to} \mathbb{A}^2$. (We write $\mathbb{A}^n$ for the formula in n variables which is always true.)

  Note that this dimension depends on K: $X = \{(x, y) \mid 2x^2 = y^2\}$ has dimension 1 if K contains $\sqrt{2}$ and 0 otherwise.
- Fix again a pseudo-finite field K. Then Chatzidakis, van den Dries and Macintyre [2] defined a *measure* μ_K. Intuitively, $\mu_K(X)$ says how "big" $X(K)$ is compared to K^d, where $d := \dim_K(X)$ is the dimension defined above. For example, $\mu_K(\mathbb{A}^n) = 1$ for any n and $\mu_K(\{\text{squares}\}) = \frac{1}{2}$. Viewing K as an ultraproduct of finite fields, one can turn the following into a precise definition of the measure:

$$\mu_K(X) := \lim_{\mathbb{F}_q \to K} \frac{\#X(\mathbb{F}_q)}{q^{\dim_K(X)}}$$

  Using μ_K, we can get many new non-bijections: For example, $\mu_K(\mathbb{A}^1) \neq \mu_K(\{\text{squares}\})$ yields $\mathbb{A}^1 \overset{1:1}{\not\to} \{\text{squares}\}$.

3. Motivic measure of Denef-Loeser

From now on and for the remaining of the talk, we will restrict ourselves to characteristic zero. Define $\mathrm{PSF}_0 := \mathrm{PSF} \cup \{\mathrm{char} = 0\}$.

Denef and Loeser [4] defined a very strong invariant for pseudo-finite fields, which I call "motivic measure". It contains the information of all dimensions and all measures defined in the previous section, and even a lot more.

The image ring of this invariant is (more or less) what is called the "Grothendieck Ring of Chow motives". I can not define this in one hour, so let me just say as much as is needed for this talk.[1]

3.1. **The Grothendieck ring of motives.**

We first need to define the Grothendieck ring of varieties (over $\mathbb{Q}$).

Definition 3. *The Grothendieck ring of varieties is* $K_0(\mathrm{Var}) := K_0(\mathrm{ACF}_0)$*, where* ACF_0 *is the theory of algebraically closed fields of characteristic zero.*

[1] The idea behind the motives is that they should form a "universal cohomology theory": to any variety V one can associate a motive which contains all the cohomological information about V.

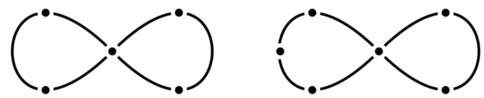

FIGURE 1. Two different decompositions of the same topological space. The Euler characteristic is $\chi = 5 - 6 = 6 - 7 = -1$.

Note that due to quantifier elimination, $K_0(\mathrm{Var})$ is indeed generated by varieties. (And in fact, usually one defines this ring using varieties.)

Now let us just say the following:

Definition 4. *The Grothendieck of motives is:* $K_0(\mathrm{Mot}) := K_0(\mathrm{Var})/$*something.*

(To be honest, this is not even true; our $K_0(\mathrm{Mot})$ is a subring of the Grothendieck ring of Chow motives; but our $K_0(\mathrm{Mot})$ is the ring we want to work with in this talk.)

3.2. **The theorem of Denef-Loeser.** The result of Denef and Loeser is that the map $K_0(\mathrm{Var}) \to K_0(\mathrm{Mot})$ can be extended to arbitrary definable sets after $K_0(\mathrm{Mot})$ has been tensored with $\mathbb{Q}$:

Theorem 5 (Denef, Loeser)**.** *There exists a (unique) ring homomorphism μ_{DL} making the following diagram commutative and having some additional properties:*

$$\begin{array}{ccc} \mu_{\mathrm{DL}}\colon K_0(\mathrm{PSF}_0) & \longrightarrow & K_0(\mathrm{Mot}) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \uparrow & & \uparrow \\ K_0(\mathrm{Var}) & \longrightarrow & K_0(\mathrm{Mot}) \end{array}$$

Now we want to apply this to check that some definable sets are not in bijection. But there is a problem: We do not know how to compare elements in $K_0(\mathrm{Mot})$. So at first sight it seems that we did not win anything. However by the very definition of the ring of motives, there are a lot of invariants of $K_0(\mathrm{Mot})$ known from algebraic geometry: all so-called "cohomological invariants", so we can compose these invariants with μ_{DL} to get a lot of useful invariants for $K_0(T)$. Let us look at an example.

3.3. **Example: the Euler characteristic.** The Euler characteristic of a variety is defined as follows:

Definition 6. *Let V be a variety. The Euler characteristic of V is defined as follows.*[2] *Consider $V(\mathbb{C})$ with the complex analytic topology. Decompose this space into a disjoint union of e_0 points, e_1 lines (subspaces homeomorphic to $\mathbb{R}$), e_2 planes, etc. Then the Euler characteristic is $\chi(V) := e_0 - e_1 + e_2 - \dots$.*

There are two lemmas hidden in this definition: any variety can be (analytically) decomposed in the above way, and the Euler characteristic does not depend on the decomposition. For an example see Figure 1.

Moreover, one can show that the Euler characteristic of a variety only depends on the corresponding motive:

[2] To be precise, we define the "Euler characteristic with compact support", i.e. in terms of cohomology: $\sum_i (-1)^i \dim H_c(V)$.

Lemma 7. *The Euler characteristic induces a ring homomorphism* $\chi\colon K_0(\mathrm{Mot}) \to \mathbb{Z}$.

Now by tensoring this with $\mathbb{Q}$ and composing with μ_{DL}, we get a map $\chi\colon K_0(T) \to \mathbb{Q}$ which associates an Euler characteristic to any definable set. This is remarkable, as the Euler characterstic was defined using the complex valued points of a variety and the analytic topology, which a priori does not make sense at all for arbitrary definable sets of the theory of pseudo-finite fields.

Application: Let $X = \{\text{squares}\}$ and $Y = \{\text{squares}\} \setminus \{0\}$. X and Y have the same dimension and measure in any pseudo-finite field, so up to now, there still might be a bijection between them. But: $\chi(X) = \chi(Y \dot\cup \{0\}) = \chi(Y) + \chi(\{0\}) = \chi(Y) + 1$. So $\chi(X) \neq \chi(Y)$, which implies $X \overset{1:1}{\not\to} Y$.

4. Generalizations

4.1. **Goals.** We will now address the following problems:

- Can this map μ_{DL} also be defined for other fields than pseudo-finite ones?
- Is it possible to get even more information about definable sets in pseudo-finite fields than using the map μ_{DL}? (Or is the map μ_{DL} already injective?)

Concerning the other fields: The main tool to get hold of definable sets is an "almost quantifier elimination" result which holds for any perfect PAC field which has only a finite number of extensions in each degree. So these fields are good candidates for a generalization. However, the map of Denef-Loeser can be seen as a motivic version of the measure μ_K of [2] (from the first section), and it turns out that in general μ_K can not be generalized to fields with larger group than $\hat{\mathbb{Z}}$; see [5]. So we do not expect to be able do generalize μ_{DL} to these fields either and restrict ourselves to fields whose Galois group is a subgroup of $\hat{\mathbb{Z}}$.

4.2. **The theorem.** The theories we will work with are the following.

Definition 8. *Let G be a torsion-free profinite procyclic group G (i.e. G is isomorphic to a subgroup of $\hat{\mathbb{Z}}$). Then denote by T_G the theory of perfect PAC fields with absolute Galois group G.*

We will be able to treat the above problems using the following theorem, which relates the Grothendieck group of theories T_G for different groups G.

Theorem 9 (see [6])**.** *Let $G_2 \subset G_1$ both be groups as above. Then the following defines a ring homomorphism $\theta\colon K_0(T_{G_2}) \to K_0(T_{G_1})$.*

Suppose X is a definable set of T_{G_2} and $K_1 \models T_{G_1}$ a model of T_{G_1}. We describe the K_1-points of $\theta(X)$:

Let $\tilde{K}_1$ be the algebraic closure of K_1. We have $\mathrm{Gal}(\tilde{K}_1/K_1) = G_1$. The subgroup G_2 defines an intermediate field K_2 which is a model of T_{G_2}. Using this, we define $\theta(X)(K_1) = X(K_2) \cap K_1^n$.

So the statement of the theorem is the following:

1. The above sets $\theta(X)(K_1)$ are definable... even uniformly in K_1.
2. This definition of θ is compatible with definable bijections, so that it induces a map $K_0(T_{G_2}) \to K_0(T_{G_1})$.
3. The map $K_0(T_{G_2}) \to K_0(T_{G_1})$ is compatible with the ring structure, i.e. θ preserves disjoint unions and products.

The third statement follows directly from the definition. The second one is not very difficult either: one can check that if ϕ defines a bijection $X \stackrel{1:1}{\to} Y$, then $\theta(\phi)$ defines a bijection $\theta(X) \stackrel{1:1}{\to} \theta(Y)$. The main work lies in the first statement.

4.3. **Application 1: μ_{DL} for other fields.** This is now easy. Suppose G is any subgroup of $\hat{\mathbb{Z}}$ and we want to define a map $K_0(T_G) \to K_0(\mathrm{Mot}) \otimes \mathbb{Q}$. Applying the theorem to $G = G_2 \subset G_1 = \hat{\mathbb{Z}}$, we get a map $\theta\colon K_0(T_G) \to K_0(T_{\hat{\mathbb{Z}}}) = K_0(T_{PSF})$ which we can compose with the map of Denef-Loeser.

Note that (in general) there are different ways to embed G into $\hat{\mathbb{Z}}$. The embedding which will make the map $K_0(T_G) \to K_0(\mathrm{Mot}) \otimes \mathbb{Q}$ most injective is the "largest" one, i.e. such that the cokernel $\hat{\mathbb{Z}}/G$ is torsion free.

4.4. **Application 2: More information about pseudo-finite fields.** It turns out that using θ, we can get even more information about pseudo-finite fields. Here is an example:

Suppose $X := \{\text{squares}\} \setminus \{0\}$ and $Y := \{\text{non-squares}\}$ (in the theory of pseudo-finite fields). By explicit computation, one can check: $\mu_{\mathrm{DL}}(X) = \mu_{\mathrm{DL}}(Y)$.

Now consider $G_2 := 2\hat{\mathbb{Z}} \subset \hat{\mathbb{Z}} =: G_1$. Both groups are isomorphic to $\hat{\mathbb{Z}}$, so applying the theorem yields an endomorphism $\theta \in \mathrm{End}(K_0(\mathrm{PSF}_0))$. We now apply this endomorphism to X and Y.

So suppose we have a pseudo-finite field K_1. We let K_2 be the extension of K_1 corresponding to $G_2 \subset G_1$, i.e. K_2 is the unique extension of K_1 of degree 2. By definition, $\theta(X)(K_1) = X(K_2) \cap K_1$ and $\theta(Y)(K_1) = Y(K_2) \cap K_1$. But the square root of any element of K_1 lies in K_2, i.e. viewed as elements of K_2, all elements of K_1 are squares. So $\theta(X)(K_1) = K_2^{\times}$ and $\theta(Y)(K_1) = \emptyset$. In particular $\theta(X) \neq \theta(Y)$, which implies $X \stackrel{1:1}{\not\to} Y$.

In particular, we found $[X] \neq [Y]$ with $\mu_{\mathrm{DL}}(X) = \mu_{\mathrm{DL}}(Y)$, which means that μ_{DL} is *not* injective.

Remark: Of course, for any fixed pseudo-finite field K there does exist a definable bijection $X(K) \to Y(K)$: multiplication by any non-square. What we just showed is that this can not be done *uniformly* without parameter for all pseudo-finite fields.

References

[1] Z. Chatzidakis, Notes on the model theory of finite and pseudo-finite fields. Notes d'un mini-cours donné à l'Université Autonome de Madrid en novembre 2005, on the web.

[2] Z. Chatzidakis, L. van den Dries, A. Macintyre, Definable sets over finite fields, J. Reine Angew. Math. 427 (1992), 107–135.

[3] Z. Chatzidakis, E. Hrushovski, Perfect pseudo-algebraically closed fields are algebraically bounded, J. Algebra 271 (2) (2004), 627–637.

[4] J. Denef, F. Loeser, Definable sets, motives and p-adic integrals, J. Amer. Math. Soc. 14 (2001), 429–469.

[5] I. Halupczok, A measure for perfect PAC fields with pro-cyclic Galois group, Journal of Algebra, 310 (2007), pp. 371–395.

[6] I. Halupczok, Motives for perfect PAC fields with pro-cyclic galois group. Preprint.