RV information of definable sets in valued fields

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Let K be a henselian valued field of characteristic $(0, 0)$. For simplicity of exposition, we assume $K = k((t))$ where k is any field of characteristic 0. Let $R = k[[t]]$ be the corresponding valuation ring. We write Γ for the value group.

It will be useful to define the valuation and the angular component of tuples $\overline{x} = \sum_{i\in\Gamma} \overline{a}_i t^i \in K^n$ (where $\overline{a}_i \in k^n$): $v(\overline{x}) := \min\{i \in \Gamma \mid \overline{a}_i \neq 0\} = \min\{v(x_1), \dots, v(x_n)\}$ and $\mathrm{ac}(\overline{x}) := \overline{a}_{v(\overline{x})}$.

For a definable set $X \subset R^n$, we want to understand the residue field and value group information (the "RV information") contained in X. More precisely, we want to describe definable sets "up to RV-isometry", which is defined as follows:

Definition 1. *A definable bijection* $f: X \to Y$ *is an* RV-isometry *if for any* $\overline{x}, \overline{x}' \in X$*, we have* $v(f(\overline{x}) - f(\overline{x}')) = v(\overline{x} - \overline{x}')$ *(i.e. it is a usual isometry) and* $\mathrm{ac}(f(\overline{x}) - f(\overline{x}')) = \mathrm{ac}(\overline{x} - \overline{x}')$.

We will present a theorem which yields a good description of definable sets up to RV-isometry; it implies that large parts of any definable set are, up to RV-isometry, translation invariant in many directions. To make this precise, we need some more definitions.

By a "ball" in R^n, we shall mean a set of the form $B = \overline{x}_0 + t^\lambda R^n = \{\overline{x} \in R^n \mid v(\overline{x} - \overline{x}_0) \geq \lambda\}$.

Call a definable set $X \subset R^n$ *translatable* on a ball B if there exists a direction $\overline{c} \in K^n \setminus \{0\}$ in which it is translation invariant on B, i.e. $(X + K\overline{c}) \cap B = X \cap B$. Call $X \subset R^n$ *almost translatable* on B if there exists an RV-isometry $X \cap B \to Y \subset B$ such that Y is translatable on B.

theorem 2. [1] *There exists a finite number of sets* S_i *each of which is either a ball or a point such that for any ball* B*,* X *is almost translatable on* B *if and only if* B *does not contain any of the sets* S_i*.*

Each ball S_i yields a finite number of balls B on which X is not almost translatable; each point S_i yields an infinite descending chain of balls. Typically, these points are singularities of X.[2]

This theorem is useful because it reduces understanding X up to RV-isometry for most of the set to lower dimension: suppose that B is a ball where X is almost translatable, i.e. $X \cap B$ is RV-isometric to a set $Y \subset B$ which is translation invariant in direction $\overline{c}$. Then Y is the preimage under π of a set $Y' \subset B'$, where $B' \subset R^{n-1}$ is a ball of the same radius as B but of lower dimension and $\pi: B \twoheadrightarrow B'$ is a suitable projection sending $\overline{c}$ to 0. Hence, up to RV-isometry $X \cap B$ is determined by Y' and the direction $\overline{c}$. Now theorem 2 can be recursively applied to Y'. In other words, we obtain that on most of the balls where X is almost translatable,

[1] "theorem" with lowercase "t" because still work in progress

[2] In a more general setting, when the value group Γ is not $\mathbb{Z}$, the theorem can still be formulated essentially in the same way. However, then even the balls S_i yield infinite chains of balls B where X is not almost translatable.

it is even RV-isometric to a set which is translation invariant in two directions, and so on.

This description is a rather strong restriction on possible RV-isometry classes of definable sets. It turns out that indeed, the number of possible RV-isometry classes which are left over is small in a precise sense: the RV-isometry class of a set can be specified using only parameters from the residue field and the value group. Moreover, this works in a definable way. More precisely:

theorem 3. *Let X_s be a definable family of definable sets ($s \in S$). Then there exists a definable map $\psi\colon S \to (k \cup \Gamma)^{\mathrm{eq}}$ such that $\psi(s) = \psi(s')$ if and only iff X_s and $X_{s'}$ are RV-isometric.*