A language for quantifier elimination in ordered abelian groups

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The goal of these notes is to present a language L_{qe} in which ordered abelian groups have a kind of quantifier elimination. This language is not that simple; the emphasis of these notes is on explaining it: why is it necessary and what does quantifier elimination in this language say about definable sets? We will build up L_{qe} piece by piece; a summary is given in the very last section.

Quantifier elimination in arbitrary ordered abelian groups was first proven by Gurevich and Schmitt in [2] and [3]. The present notes are based on a new proof in [1] by Cluckers and the author. The language L_{qe} (denoted by L_{int} in [1]) is somewhat different from the one used by Gurevich and Schmitt; however, translation between the different languages is not very difficult.

1 Ordered abelian groups with finitely many convex definable subgroups

1.1 Basic notation

Ordered abelian groups are abelian groups endowed with an order compatible to the group structure: $a < a' \Rightarrow a + b < a' + b$; they are naturally first order structures in the language $L_{\mathrm{oag}} = \{0, +, -, <\}$. One easily checks that such groups are always torsion-free. Standard examples of ordered abelian groups are the additive groups of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, and the multiplicative groups of $\mathbb{Q}_{>0}$ and $\mathbb{R}_{>0}$. Moreover, ordered abelian group arise naturally as valuation groups of valued fields.

In the whole article, G will be an ordered abelian group. At some point, we will extend the language by new ("auxiliary") sorts; in that setting G will denote the group sort.

"Definable" means definable with parameters.

We will write $\mathbb{P}$ for the set of primes.

1.2 Ordered abelian groups without convex definable subgroup

It is well known that the ordered abelian group $\mathbb{Z}$ has quantifier elimination in the Presburger language $L_{\text{Pres}} := \{0, 1, +, -, <, (\equiv_m)_{m\in\mathbb{N}}\}$; here, $x \equiv_m y$ is a binary predicate defined by $x \equiv y \mod m$. Of course, $x \equiv_m y$ could have been replaced by a unary predicate for mG, and there is no real reason to prefer one or the other; we decided to prefer $x \equiv_m y$.

The same language yields quantifier elimination in any fixed ordered abelian group G which has no convex definable subgroup; we only have to generalize 1 and $\equiv_m$ correctly: 1 is the minimal positive element of G, or 0 if G has no minimal positive element (we say that G is *discrete* if a minimal positive element exists and *dense* otherwise), and $x \equiv_m y$ is defined by $x - y \in mG$. So:

Theorem 1.1 *Any (fixed) ordered abelian group without convex definable subgroup has quantifier elimination in L_{Pres}.*

The first examples one thinks of are probably subgroups of $(\mathbb{Q}, +)$. However, this might give the false impression that the cardinality of G/mG is at most m. A counter-example to this is $(\mathbb{Q}_{>0}, \cdot)$, which has no convex definable subgroup, but for any $m \geq 2$, $\mathbb{Q}_{>0}/\{m\text{-th powers in } \mathbb{Q}_{>0}\}$ is infinite.

1.3 Ordered abelian groups with finitely many convex definable subgroups

To get a language in which arbitrary ordered abelian groups have quantifier elimination, the main work is to treat convex definable subgroups. Let us already treat an easy case, namely that of a fixed group G which moreover has only finitely many convex definable subgroups. For the moment let us assume that G has only one non-trivial convex definable subgroup H; as an example, you can think of $G = \mathbb{Z} \oplus \mathbb{Z}$ with the lexicographical order; we will see below (in Section 2.1) that the convex subgroup $H := \{0\} \times \mathbb{Z}$ is indeed definable.

Essentially, the language we need for quantifier elimination is L_{Pres} on G, together with L_{Pres} on the quotient G/H. This could be achieved using a new sort for G/H, together with the canonical map $\pi\colon G \twoheadrightarrow G/H$, but such a new sort will be unpractical when we will have a lot of convex definable subgroups; so instead, we "lift" L_{Pres} from G/H to G, i.e. we choose a language on G which allows us to define the preimage of the Presburger relations on G/H.

Lifting the congruence condition $\equiv_m$ from G/H to G is easy: define $a \equiv_{m,H} b$ (for $a, b \in G$) by $\pi(a) - \pi(b) \in m \cdot (G/H)$ (or, equivalently: $a - b \in H + mG$). Lifting the constant for the minimal positive element of G/H (let me denote it by $1_{G/H}$) is more problematic: we can not take a constant in G, since $\pi^{-1}(1_{G/H})$ is not just a single element and it does not contain a canonical (i.e. 0-definable) representant. Adding a predicate for $\pi^{-1}(1_{G/H})$ instead is not enough, since we need to be able to build terms using this "element", and we need to plug it into the different L_{Pres} relations. The solution is to add predicates for the lifts of (essentially) every atomic L_{Pres}-formula:

- for each $k \in \mathbb{Z}$, take a binary predicate on G denoted by "$x =_H y + k_{G/H}$" and defined by $\pi(x) = \pi(y) + k_{G/H}$; here, $k_{G/H}$ denotes k times the minimal positive element of G/H.
- for each $k \in \mathbb{Z}$, and each $m \in \mathbb{N}$, take a binary predicate "$x \equiv_{m,H} y + k_{G/H}$" on G defined by $\pi(x) \equiv_m \pi(y) + k_{G/H}$.

(We will sometimes use simplified notations "$x =_H y$", "$x \equiv_{m,H} y$", "$x =_H k_{G/H}$", and "$x \equiv_{m,H} k_{G/H}$" with the obvious meanings.)

Note that we do not need predicates for $\pi(x) > \pi(y) + k_{G/H}$, since for example, we have

$$\pi(x) > \pi(y) + 1_{G/H} \iff x > y \ \wedge \ x \neq_H y \ \wedge \ x \neq_H y + 1_{G/H}.$$

With this language, we get quantifier elimination. Moreover, a similar construction works with any finite number of convex definable subgroups:

Theorem 1.2 *Suppose G is an ordered abelian group which has only finitely many convex definable subgroups $H_1, \ldots, H_\nu$ (including $\{0\}$ but excluding G). Then G has quantifier elimination in the language*

$$L_{\text{oag}} \cup \{(x =_{H_i} y + k_{G/H_i})_{k \in \mathbb{Z}, i \leq \nu}, (x \equiv_{m,H_i} y + k_{G/H_i}))_{k \in \mathbb{Z}, m \in \mathbb{N}, i \leq \nu}\}.$$

Note that to make the language more uniform, we removed $\equiv_m$ and 1 from the language and instead added $x =_H y + k_{G/H}$ and $x \equiv_{m,H} y + k_{G/H}$ for $H = \{0\}$.

Also note that in this setting, any convex definable group is already 0-definable, since there are only finitely many of them and they are ordered by inclusion. This means that the language of Theorem 1.2 is indeed L_{oag}-definable.

2 Many convex definable subgroups and relative quantifier elimination

In general, an ordered abelian group has infinitely many infinite families of convex definable subgroups. We will not need to add the relations $x =_H y + k_{G/H}$ and $x \equiv_{m,H} y + k_{G/H}$ for all of them, but at least for some of these infinite families. Since this means that no fixed number of relations would be enough, we will introduce canonical parameters for those convex subgroups we are interested in and define the above relations as ternary relations. Before going into details, let me give some examples of groups with many convex definable subgroups, and before giving examples, I have to present one way of defining convex definable subgroups.

2.1 Defining convex subgroups

For a set $X \subseteq G \setminus \{0\}$, we can consider the largest convex subgroup $H \subseteq G$ not meeting X. Another way of writing this is: $H = \{h \in G \mid \langle h \rangle^{\text{conv}} \cap X = \emptyset\}$, where $\langle h \rangle^{\text{conv}}$ is the smallest convex subgroup of G containing h. In general, H

is not definable (since $\langle h\rangle^{\mathrm{conv}}$ is not definable), but if X is of the form $a + nG$ (for $a \in G$ and $n \in \mathbb{N}$), then we can define H as follows. First note that $\langle h\rangle^{\mathrm{conv}} \cap (a + nG) = \emptyset$ if and only if $a \notin \langle h\rangle^{\mathrm{conv}} + nG$. Now, using the notation $|h|$ for the absolute value of h, we have $\langle h\rangle^{\mathrm{conv}} + nG = \{x \mid 0 \leq x \leq n \cdot |h|\} + nG$, which is definable.

We will need this construction quite often, so let us denote the largest convex subgroup not meeting $a + nG$ by $H_n(a)$. To make this well-defined for all $a \in G$, we set $H_n(a) = \{0\}$ if $a \in nG$.

In the example $G = \mathbb{Z} \oplus \mathbb{Z}$ from Section 1.3, $H := \{0\} \times \mathbb{Z}$ is the maximal convex subgroup not meeting $(1, 0) + 2G = (1 + 2\mathbb{Z}) \times 2\mathbb{Z}$, so it can be defined as $H_2((1, 0))$. More generally, one easily checks that for any $n \in \mathbb{N}$ and $(a, b) \in G$, $H_n((a, b))$ is $\{0\}$ if $a \in n\mathbb{Z}$ and H otherwise.

2.2 Many convex definable subgroups: an example

Now let me give an example of an ordered abelian group with many convex definable subgroups. Let I be any ordered set and consider the group $G = \bigoplus_{i \in I} \mathbb{Z}$ with lexicographical order "with significance according to I". More precisely, for $a = (a_i)_{i \in I} \in G$, let us denote by $v(a) := \max\{i \in I \mid a_i \neq 0\}$ the largest index of a non-zero entry of a (set $v(0) := -\infty$); this is well-defined, since only finitely many a_i are non-zero. Now define the order on G by $a > 0$ iff $a \neq 0$ and $a_{v(a)} > 0$.

Before going on, let me introduce some more notation:

- for $j \in I$ let $g_j : \mathbb{Z} \hookrightarrow G$ be the map sending $\mathbb{Z}$ to the j-th summand of G;
- for $j \in I$, define two convex subgroups: $G_{<j} := \{g \in G \mid v(g) < j\}$ and $G_{\leq j} := \{g \in G \mid v(g) \leq j\}$ (or equivalently, in a sloppy notation: $G_{<j} := \bigoplus_{i<j} \mathbb{Z}$ and $G_{\leq j} := \bigoplus_{i \leq j} \mathbb{Z}$).

All groups $G_{<j}$ are definable: $G_{<j} = H_n(g_j(1))$ for any $n \geq 2$. Moreover, we can check that for any $a \in G \setminus nG$, $H_n(a)$ is of the form $G_{<j}$ for some $j \in I$. Indeed, if $a = (a_i)_{i \in I}$ and j is the most significant (i.e. largest) index such that $a_j \notin n\mathbb{Z}$, then $\min\{v(x) \mid x \in a + nG\} = j$ and we obtain $H_n(a) = G_{<j}$. Thus the groups $G_{<j}$ form an infinite family of convex uniformly definable subgroups, defined by $(H_n(a))_{a \in G \setminus nG}$ (for any fixed $n \geq 2$). In particular I can be interpreted in G: take the quotient of $G \setminus nG$ by the equivalence relation $a \sim b$ iff $H_n(a) = H_n(b)$. The order relation of I can be recovered on $G/{\sim}$ using the inclusion of the corresponding groups $H_n(a)$.

Being able to interpret arbitrary ordered sets in (suitable) ordered abelian groups is a serious problem concerning quantifier elimination: ordered sets may have too complicated definable subsets (but see the last remark of Section 2.5). This means that in ordered abelian groups, the best one can hope for is "quantifier elimination up to ordered sets", and indeed, this is what we will get. The meaning of this will be explained in Section 2.5.

2.3 Different families of convex definable subgroups

In the previous example, the family of convex definable subgroups defined by $H_n(x)$ did not depend on n (as long as $n \geq 2$). In general, the families may very well depend on n. A small example is $G = \mathbb{Z}[\frac{1}{5}] \oplus \mathbb{Z}$. As before, we have $H_2((1,0)) = \{0\} \times \mathbb{Z} =: H$. However, now H is not of the form $H_5((a,b))$ for any $(a,b) \in G$, since 5-divisibility of $\mathbb{Z}[\frac{1}{5}]$ implies that $(a,b) + 5G = \mathbb{Z}[\frac{1}{5}] \times (b + 5\mathbb{Z})$, which always meets H.

Thus to obtain all groups $H_n(a)$, it is not enough to consider only one fixed n. On the other hand, one can check that any group $H_n(a)$ is equal to $H_p(a')$ for a suitable prime p (and some $a' \in G$ possibly different from a). In our example, this is easy to see. We can replace $\mathbb{Z}[\frac{1}{5}]$ by any group of the form $\mathbb{Z}[\frac{1}{p}]_{p \in A}$ where $A \subseteq \mathbb{P}$ is any set of primes. In that case, H appears in the family $H_n(x)$ if and only if A does not contain all prime factors of n; in particular $H = H_p((1,0))$ for any $p \in \mathbb{P} \setminus A$. If, on the other hand, $A = \mathbb{P}$, then H is not of the form $H_n(a)$ for any n and even not definable at all (and $G = \mathbb{Q} \times \mathbb{Z}$ is elementarily equivalent to $\mathbb{Z}$).

2.4 A language which can cope with families of convex definable subgroups

In the quantifier elimination language L_{qe}, we would like to have our relations $x =_H y + k_{G/H}$ and $x \equiv_{m,H} y + k_{G/H}$ (from Section 1.3) for any group H of the form $H_n(a)$ (but without loss, n prime). To this end, for each $p \in \mathbb{P}$ we introduce a new sort $\mathcal{S}_p$ with canonical parameters for the groups $H_p(x)$: we define $\mathcal{S}_p$ to be the quotient of G by the equivalence relation $\sim$ defined by $a \sim b$ iff $H_p(a) = H_p(b)$. Let us write $\mathfrak{s}_p \colon G \twoheadrightarrow \mathcal{S}_p$ for the canonical map, and for $\alpha = \mathfrak{s}_p(a) \in \mathcal{S}_p$, let us write $G_\alpha := H_p(a)$ for the corresponding group.

Now we define $x =_{G_\alpha} y + k_{G/G_\alpha}$ and $x \equiv_{m,G_\alpha} y + k_{G/G_\alpha}$ as before, but we view them as ternary relations on $G \times G \times \mathcal{S}_p$. To be more precise, we have one such set of relations for each $p \in \mathbb{P}$. Let us simplify the notation and write $x =_\alpha y + k_\alpha$ and $x \equiv_{m,\alpha} y + k_\alpha$.

In our example $G = \bigoplus_{i \in I} \mathbb{Z}$, we already computed $\mathcal{S}_p$: it is essentially just I (only "essentially" because we defined $H_p(a)$ to be the trivial group if $a \in pG$; this trivial group might yield an additional element in $\mathcal{S}_p$). Moreover, we obtained that $G_\alpha = G_{<j}$ if $\alpha \in \mathcal{S}_p$ corresponds to $j \in I$.

2.5 The result

In the final L_{qe}, we will have the group sort G and some "auxiliary" sorts with canonical parameters for convex definable subgroups: the sorts $\mathcal{S}_p$ which we already defined and some other ones which will be defined in Section 3.1. As already mentioned, we will only eliminate quantifiers "up to ordered sets", or, more precisely, "up to the auxiliary sorts"; now let me explain what this should mean.

Let us write L_{qe} as a union $L_{\text{qe-grp}} \cup L_{\text{qe-aux}}$, where $L_{\text{qe-grp}}$ consists of the part of the language involving the group sort, whereas $L_{\text{qe-aux}}$ is the part of the language living purely on the auxiliary sorts.

The language $L_{\text{qe-grp}}$ will be

$$L_{\text{qe-grp}} = L_{\text{oag}} \cup \{x =_\alpha y + k_\alpha,\ x \equiv_{m,\alpha} y + k_\alpha,\ x \equiv^{[n]}_{m,\alpha} y\}$$

where $x =_\alpha y + k_\alpha$ and $x \equiv_{m,\alpha} y + k_\alpha$ are the ternary relations defined in Section 2.4, but extended to all auxiliary sorts, and where $x \equiv^{[n]}_{m,\alpha} y$ are some additional ternary relations similar to $x \equiv_{m,\alpha} y$ which will be introduced later. This implies: if $\psi(\bar{x}, \bar{\alpha})$ is a quantifier-free $L_{\text{qe-grp}}$-formula with $\bar{x}$ a tuple of group variables and $\bar{\alpha} = (\alpha_1, \dots, \alpha_\nu)$ a fixed tuple of auxiliary elements, then the set $\psi(G, \bar{\alpha})$ is of almost the same form as the definable sets in the case of finitely many convex definable subgroups $G_{\alpha_1}, \dots G_{\alpha_\nu}$ ("almost" because of $x \equiv^{[n]}_{m,\alpha} y$).

We can not expect that arbitrary definable sets can be defined like this; a typical example of a set involving infinitely many convex definable subgroups in its definition is $X := \{x \in G \mid \exists(\alpha \in \mathcal{S}_p)\, x =_\alpha 1_\alpha\}$. However, X is the union of a uniform family of disjoint quantifier free $L_{\text{qe-grp}}$-definable sets: $\{x \in G \mid x =_\alpha 1_\alpha\}$, where α runs over $\mathcal{S}_p$. Our relative quantifier elimination result states that any definable set is a disjoint union of finitely many such families.

In the case of X, the parameter of the family runs over $\mathcal{S}_p$; in general, the parameter might run over any definable set Ξ in the auxiliary sorts, and since the induced structure on these sorts can (at least) be the one of arbitrary ordered sets, we can not hope to describe Ξ very precisely. However, at least we obtain that the induced structure can (essentially) not be more complicated than that of an ordered set. More precisely, we will get that Ξ is $L_{\text{qe-aux}}$-definable, and $L_{\text{qe-aux}}$ will be

$$L_{\text{qe-grp}} = \{<, \text{some unary predicates}\},$$

where $<$ is the preorder on the union of all auxiliary sorts defined by $\alpha < \alpha'$ iff $G_\alpha \subsetneq G_{\alpha'}$. (Formally, there is one relation $<$ for each pair of auxiliary sorts; of course, on each single auxiliary sort $<$ is actually an order, but if α and α' come from different sorts, G_α and $G_{\alpha'}$ might be equal.)

Here is the complete formulation of the relative quantifier elimination result:

Theorem 2.1 *Let $\phi(\bar{x}, \bar{\eta})$ be a L_{qe}-formula, where $\bar{x}$ are group-variables and $\bar{\eta}$ are variables from any of the auxiliary sorts. Modulo the theory T_{oag} of ordered abelian groups, ϕ is equivalent to a formula of the following form:*

$$\bigvee_{i=1}^{k} \exists \bar{\theta} \left(\xi_i(\bar{\eta}, \bar{\theta}) \wedge \psi_i(\bar{x}, \bar{\theta}) \right);$$

here, $\bar{\theta}$ are auxiliary variables, $\xi_i(\bar{\eta}, \bar{\theta})$ are $L_{\text{qe-aux}}$-formulas, and $\psi_i(\bar{x}, \bar{\theta})$ are quantifier free $L_{\text{qe-grp}}$-formulas. Moreover, all sets $\{(\bar{x}, \bar{\eta}) \mid \xi_i(\bar{\eta}, \bar{\theta}) \wedge \psi_i(\bar{x}, \bar{\theta})\}$ (for different $i, \bar{\theta}$) are disjoint.

We say that a formula of the form given in the theorem is in "family union form".

Some remarks:

- If one wants, one can easily turn each $\psi_i(\bar{x}, \bar{\theta})$ into a conjunction of literals (by moving disjunctions to the exterior).
- As in the case of finitely many definable convex subgroups, 1 and $\equiv_n$ are not necessary in the language, since the trivial convex group appears in our families as $H_p(0)$.
- Note that the canonical maps $\mathfrak{s}_p$ are not needed either; indeed, they would destroy some of the simplicity of the formulas ψ_i.
- In ordered sets (with or without additional unary predicates) one can not hope for quantifier elimination, but still there are some nice model theoretic results: definable subsets of cartesian powers have a simple description in terms of definable subsets of the ordered set itself, and moreover ordered sets have NIP. Gurevich and Schmitt used this to deduce that ordered abelian groups have NIP, too.
- Theorem 1.1 (on groups without convex definable subgroup) was valid only in one fixed group, since the theory of all such groups is not complete, and distinguishing different completions needs quantifiers. In L_{qe}, this distinction will be encoded in the unary predicates on the auxiliary sorts (which, in the setting of Theorem 1.1, consists only of one element corresponding to the trivial group); see Section 3.7.

3 The remainder of the language

3.1 Two more families of families of convex subgroups

Up to now, we saw one family of families of convex definable subgroups. We will need two more such families of families in our language; these will be constructed out of the one we already have.

Fix one sort $\mathcal{S}_p$. For any element $a \in G$, we can look at all those groups G_α ($\alpha \in \mathcal{S}_p$) which contain a. This allows us to define two new convex definable groups:

$$H_p^-(a) := \bigcup_{\{\alpha \in \mathcal{S}_p \mid a \notin G_\alpha\}} G_\alpha \quad \text{and} \quad H_p^+(a) := \bigcap_{\{\alpha \in \mathcal{S}_p \mid a \in G_\alpha\}} G_\alpha. \qquad (*)$$

We define two new auxiliary sorts $\mathcal{T}_p$ and $\mathcal{T}_p^+$ with canonical parameters for $H_p^-(a)$ and $H_p^+(a)$ in the same way as we did for $H_p(a)$ (as quotients of G by the obvious equivalence relations). Then we extend the ternary relations $x =_\alpha y + k_\alpha$ and $x \equiv_{m,\alpha} y + k_\alpha$ and the preorder $<$ on $\dot{\bigcup}_p \mathcal{S}_p$ to these new sorts.

Let us also extend the notation G_α (for the group corresponding to α) to the new sorts, and let us write $\mathfrak{t}_p \colon G \twoheadrightarrow \mathcal{T}_p$ for the canonical map. Instead of also naming the canonical map $G \twoheadrightarrow \mathcal{T}_p^+$, note the following. Both, $H_p^-(a)$ and $H_p^+(a)$, only depend on the set of groups G_α ($\alpha \in \mathcal{S}_p$) containing a, so formally, $\mathcal{T}_p$ and $\mathcal{T}_p^+$ are quotients of G by the same equivalence relation. In particular, we get a canonical bijection $\mathcal{T}_p \to \mathcal{T}_p^+, \alpha \mapsto \alpha+$, and the canonical map $G \twoheadrightarrow \mathcal{T}_p^+$ can be written as $a \mapsto \mathfrak{t}_p(a)+$. Having separate sorts $\mathcal{T}_p$ and $\mathcal{T}_p^+$ is useful nevertheless for the notation G_α and to get a well-defined preorder on the union of all auxiliary sorts.

As in the case of $H_p(a)$, we have to decide how to define $H_p^-(a)$ and $H_p^+(a)$ in some extremal cases (when the big unions/intersections in (*) run over the empty set); this introduces a little bit of technical uglyness but no serious problems, so I won't go into the details.

3.2 The new families and an old example

Now let me give some examples for these new convex subgroups.

In our example $G = \bigoplus_{i \in I} \mathbb{Z}$ of Section 2.2, for $a \in G$ and $\alpha \in \mathcal{S}_p$ we have $a \in G_\alpha$ iff $v(a) < i$, where $i \in I$ corresponds to α; hence

$$H_p^-(a) = \bigcup_{i \le v(a)} G_{<i} = G_{<v(a)} \quad \text{and} \quad H_p^+(a) = \bigcap_{i > v(a)} G_{<i} = G_{\le v(a)}.$$

Thus as $\mathcal{S}_p$, the sorts $\mathcal{T}_p$ and $\mathcal{T}_p^+$ can be identified with I (again possibly except for extremal elements) and under this identification, the canonical map $\mathfrak{t}_p \colon G \twoheadrightarrow \mathcal{T}_p$ is simply our map v. Moreover, the family $H_p^-(a)$ consists of the same groups as $H_p(a)$ (but depends on a in a different way).

3.3 The new families and a new example

Now let us consider a (for simplicity very concrete) example where the groups $H_p^-(a)$ are also new ones, and where moreover $\mathcal{T}_p$ and $\mathcal{T}_p^+$ are really different from $\mathcal{S}_p$. We let G be a lexicographic sum of some copies of $\mathbb{Z}$ and some copies of $\mathbb{Q}$, namely

$$G := \bigoplus_{i \in \mathbb{R}} \begin{cases} \mathbb{Z} & \text{if } i \in \mathbb{Q} \\ \mathbb{Q} & \text{if } i \notin \mathbb{Q}, \end{cases}$$

with significance according to the usual order on $\mathbb{R}$. As before, we let $v(a) \in \mathbb{R}$ be the index of the most significant non-zero component of $a \in G$, and for $j \in \mathbb{R}$, we write g_j for the inclusion of the j-th summand ($\mathbb{Z}$ or $\mathbb{Q}$) into G. Moreover, we again set $G_{<j} := \{a \in G \mid v(a) < j\}$ and $G_{\le j} := \{a \in G \mid v(a) \le j\}$.

Fix $p \in \mathbb{P}$ and consider $a := g_j(1)$ for some $j \in \mathbb{R}$. If $j \in \mathbb{Q}$, then as before we obtain $H_p(a) = G_{<j}$. However, if $j \notin \mathbb{Q}$, then $a \in pG$, so $H_p(a) = \{0\}$. More generally, it is not difficult to see that $\mathcal{S}_p$ is essentially equal to $\mathbb{Q}$.

Now consider $H_p^-(a)$ (still for $a := g_j(1)$). It is the union of all $G_{<i}$ with $i < j$, so we obtain $H_p^-(a) = G_{<j}$, independently of whether $j \in \mathbb{Q}$ or not; in particular, each $j \in \mathbb{R}$ yields a different group $G_{<j}$. Instead of $a = g_j(1)$, we could as well have taken any element $a \in G$ with $v(a) = j$, so we obtain that $\mathcal{T}_p$

is essentially equal to $\mathbb{R}$. Similar arguments yield $H_p^+(a) = G_{\leq j}$ and that $\mathcal{T}_p^+$ is essentially equal to $\mathbb{R}$, too.

3.4 Defining $\mathfrak{t}_p$ in L_{qe}

In both examples, we saw that for $\alpha \in \mathcal{T}_p$, we have $G_\alpha = \{x \in G \mid \mathfrak{t}_p(x) < \alpha\}$ and $G_{\alpha+} = \{x \in G \mid \mathfrak{t}_p(x) \leq \alpha\}$; this can easily be verified in general. Using that the map $\alpha \mapsto \alpha+$ is $L_{\text{qe-aux}}$-definable ($\alpha+$ is the minimal element of $\mathcal{T}_p^+$ which is larger than α), this yields a way to define $\mathfrak{t}_p$ by a formula in family union form:

$$\mathfrak{t}_p(x) = \alpha \iff \exists(\beta \in \mathcal{T}_p^+)\ \underbrace{\beta = \alpha+}_{\xi_1} \ \wedge\ \underbrace{x =_\beta 0 \wedge x \neq_\alpha 0}_{\psi_1}.$$

This allows us to use literals of the form "$\mathfrak{t}_p(x) = \alpha$" in the ψ_i of formulas in family union form; see the example in the next section.

3.5 Even more convex definable subgroups

We added canonical parameters for the groups $H_p(x)$, $H_p^-(x)$, and $H_p^+(x)$ to the language. In general, there are many other convex definable subgroups, but but we don't need more language. Any convex definable subgroup of G can be defined in the following way: choose a prime p, and choose a definable initial segment $\Xi \subseteq \mathcal{T}_p$ (initial segment means: if $\alpha \in \Xi$ and $\alpha' < \alpha$ then $\alpha' \in \Xi$). This yields the convex definable subgroup $H := \{x \in G \mid \mathfrak{t}_p(x) \in \Xi\}$. In general, H is not in any of our families. However, we have

$$x \in H \iff \exists(\theta \in \mathcal{T}_p)\big(\theta \in \Xi \ \wedge\ \mathfrak{t}_p(x) = \theta\big),$$

which can be brought into family union form using the previous section:

$$\exists(\theta \in \mathcal{T}_p, \theta' \in \mathcal{T}_p^+)\big(\underbrace{\theta \in \Xi \wedge \theta' = \theta+}_{\xi_1} \ \wedge\ \underbrace{x =_{\theta'} 0 \wedge x \neq_\theta 0}_{\psi_1}\big).$$

(Note that $x \neq_\theta 0$ is only necessary to make the family disjoint.)

3.6 The last piece of $L_{\text{qe-grp}}$

There is one kind of subgroup of G which we can not yet define by a formula in family union form: let α be any auxiliary element, and consider the group

$$G_\alpha^{[n]} := \bigcap_{\substack{H \subseteq G \text{ conv. subgrp} \\ H \supsetneq G_\alpha}} (H + nG).$$

A priori, it is not clear that $G_\alpha^{[n]}$ is definable at all; however, it is not difficult to check that it is equal to $\{x \in G \mid H_n(x) \subseteq G_\alpha\}$. (Recall that even if n is not prime, $H_n(x)$ is definable in G.)

The problem with writing $G_\alpha^{[n]}$ in family union form is that it might not be equal to $H_0 + nG$ for any convex subgroup $H_0 \subseteq G$; an example is given at the end of [1]. The only solution is to add a new relation to $L_{\text{qe-grp}}$ defining $G_\alpha^{[n]}$. However, this alone is not enough, since we also need to be able to write the groups $G_\alpha^{[n]} + mG$ in family union form. (If m is a multiple of n, then this is equal to $G_\alpha^{[n]}$, but each m dividing n yields a new group.) Thus, for each $n \in \mathbb{N}$ and each m dividing n, we add to $L_{\text{qe-grp}}$ the ternary relation $x \equiv_{m,\alpha}^{[n]} y$ defined by $x - y \in G_\alpha^{[n]} + mG$. This finishes the definition of $L_{\text{qe-grp}}$.

3.7 Unary predicates on the auxiliary sorts

Now the only missing part of the language L_{qe} are some unary predicates on the auxiliary sorts (see Section 2.5). These predicates will reflect definable properties of the corresponding quotient groups G/G_α which otherwise would need group variables (and group quantifiers) in the definition. For all of them, it turns out that one only needs them on the sorts $\mathcal{S}_p$ (and not on $\mathcal{T}_p$ and $\mathcal{T}_p^+$).

The first observation is that G/G_α might be discrete or dense, so we add a predicate for this. (This predicate is only needed on $\mathcal{S}_p$ because whenever G/G_α is discrete, G_α appears in the family $H_p(x)$—even for every $p \in \mathbb{P}$.)

The remaining predicates concern cardinalities of certain quotient groups. In the example of Section 2.2, if α and α' are two consecutive elements of $\mathcal{S}_p$, then the quotient $(G_{\alpha'} + nG)/(G_\alpha + nG)$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. However, such successive quotients might also be larger and even infinite (recall the example $G = (\mathbb{Q}_{>0}, \cdot)$ right after Theorem 1.1). This yields predicates on $\mathcal{S}_p$ which we define to hold for α iff the corresponding quotient has a certain given cardinality. Moreover, similar quotients can be build using the groups $G_\alpha^{[n]} + mG$. Here is the complete list of cardinality predicates we ultimately need to add to $L_{\text{qe-aux}}$.

For $\alpha \in \mathcal{S}_p$, let Q_α be one of the groups

$$(G_\alpha^{[n']} + m)/(G_\alpha^{[n'']} + mG) \quad \text{and} \quad (G_\alpha^{[n']} + mG)/(G_\alpha + mG)$$

for some $m \mid n' \mid n''$. For each choice of Q_α and each $\ell \in \mathbb{N}$, we put one predicate on $\mathcal{S}_p$ saying that the cardinality of Q_α is ℓ.

Before, I promised that we would add a predicate for $Q_\alpha = (G_{\alpha'} + nG)/(G_\alpha + nG)$ when α' is a successor of α. Indeed, if α has a successor α', then we have $G_\alpha^{[n]} = G_{\alpha'} + nG$, so $(G_\alpha^{[n]} + nG)/(G_\alpha + nG)$ is exactly this group.

3.8 Summary of the language L_{qe}

Let me finish by summarizing L_{qe}. The sorts are the following:

- one sort G for the group
- for each $p \in \mathbb{P}$, three auxiliary sorts: $\mathcal{S}_p$ (see Section 2.4), $\mathcal{T}_p$, and $\mathcal{T}_p^+$ (see Section 3.1).

$L_{\text{qe-grp}}$ (the part of L_{qe} involving G) consists of the following:

- $L_{\text{oag}} = \{0, +, -, <\}$
- for each $p \in \mathbb{P}$, each $k \in \mathbb{Z}$, and each $m, n \in \mathbb{N}$ with $m \mid n$, the following ternary relations for $\alpha \in \mathcal{S}_p \mathbin{\dot\cup} \mathcal{T}_p \mathbin{\dot\cup} \mathcal{T}_p^+$: $x =_\alpha y + k_\alpha$, $x \equiv_{m,\alpha} y + k_\alpha$ (see Sections 1.3 and 2.4), and $x \equiv_{m,\alpha}^{[n]} y$ (see Section 3.6).

$L_{\text{qe-aux}}$ (the part of L_{qe} living entirely on the auxiliary sorts) consists of the following:

- for on each pair of auxiliary sorts, a binary relation $<$ (see Section 2.5)
- for each $p \in \mathbb{P}$, the unary predicates on $\mathcal{S}_p$ listed in Section 3.7

References

[1] Raf Cluckers and Immanuel Halupczok, *Quantifier elimination in ordered abelian groups*, Work in progress.

[2] Y. Gurevic, *The decision problem for some algebraic theories*, 1968, Doctor of Mathematics dissertation.

[3] P. H. Schmitt, *Model theory of ordered abelian groups*, 1982, Habilitationsschrift.